Signals & Systems

Lecture 31

Difference Equation and Z-Transform

John T. Wen

Today:

- Solution of difference equation
- Examples Ref: O&W 2.4, 10.1-10.6

April 17, 2002

Difference Equation

Solution of difference equation is the same as for differential equation:

- Find the homogeneous solution, $y_h[n]$, by using the trial solution of the form z^n.
- Find the particular solution, $y_p[n]$, for the step input $u[n]$.
- Step response is the sum of homogeneous and particular solutions: $s[n] = y_h[n] + y_p[n]$. Use the initial condition to resolve the unknown constants in $y_h[n]$.
- Find the impulse response $h[n]$ from the step response: $h[n] = s[n] - s[n-1]$.
- General solution with input $x[n]$: sum of zero input response and zero state response, $y[n] = y_{zi}[n] + y_{zs}[n]$. $y_{zi}[n]$ is the same as $y_h[n]$ with constants resolved by using the given initial condition. $y_{zs}[n]$ is given by the convolution sum: $y_{zs}[n] = h[n] * x[n]$.

Example

Find the general solution of

$$y[n] + 3y[n-1] = x[n]$$

Z Transform

Similar to using Laplace transform to find time response of continuous time linear time invariant systems, we can use z transform to find the time response of discrete time linear time invariant systems.

We will perform algebraic operations instead of solving difference equations and applying discrete convolutions.

<u>Bilateral z Transform</u>
Given a discrete-time signal $x[n]$

$$X(z) = Z\{x(t)\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$= \cdots + x[-2]z^2 + x[-1]z^1 + x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$$

where z is a complex variable.

Same region of convergence (ROC) consideration as before, except right/left half planes in s plane becomes exterior/interior of circles in z plane.

Examples

1. Find $X(z)$ for $x[n]$ given by

$$x[0] = 5, x[1] = 2, x[2] = 4$$

$$x[n] = 0 \quad \text{for other values of } n$$

2. Find $x[n]$ given

$$X(z) = 3 + 5z^{-1} + 6z^{-2} + 2z^{-3}$$

3. Find $Z\{\delta[n]\}$

What is the advantage of Z transform?

For signals in the form of a power function, e.g.,

$$x[n] = a^n u[n], \quad n = \ldots, 0, 1, 2, 3, \ldots$$

$X(z)$ can be expressed in closed form using infinite series summation.

Example: $x[n] = a^n u[n] = [1, a, a^2, a^3, \cdots]$

Claim: If $|z| > |a|$, then $X(z)$ converges and can be expressed as

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

Inverse Z transform

Given an $X(z)$, how to find $x[n]$?

Two methods:

- Long division
- Partial fraction expansion

Examples:

$$X(z) = \frac{3-5/6z^{-1}}{(1-.25z^{-1})(1-1/3z^{-1})}$$

Properties of Z Transform

- Linearity/Superposition

$$x_1[n] \overset{Z}{\leftrightarrow} X_1[n], \quad x_2[n] \overset{Z}{\leftrightarrow} X_2[n]$$

Then for a, b scalars

$ax_1[n] + bx_2[n] \overset{Z}{\leftrightarrow}$

- Time shifting

$$x[n] \overset{Z}{\leftrightarrow} X(z)$$

then

$$x[n - n_o] \overset{Z}{\leftrightarrow} z^{-n_o} X(z)$$

Another example:

$$X(z) = \frac{z+1}{z-0.5}$$

More Properties

- Time reversal

$$x[-n] \stackrel{Z}{\leftrightarrow} X(\frac{1}{z})$$

- Convolution

$$x_1[n] * x_2[n] \stackrel{Z}{\leftrightarrow} X_1[n]X_2[n]$$

- Differentiation in z domain

$$nx[n] \stackrel{Z}{\leftrightarrow} -z\frac{dX(z)}{dz}$$

Initial and Final Value Theorems

Final Value Theorem - asymptotically stable signal or system

$$\lim_{n\to\infty} x[n] = \lim_{z\to 1}(z-1)X(z)$$

Example: $x[n] = (3-0.2^n)u[n]$

Initial Value Theorem for signals that are 0 when $n<0$

$$x[0] = \lim_{z\to\infty} X(z)$$

Example: the signal $x[n]$ with $x[0]=5$, $x[1]=4$, $x[2]=1$, and $x[n]=0$ for all other n including $n<0$